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NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

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[Continued from March Number.]

PROPOSITION XXXIV, *in which is investigated a certain curve arising from the hypothesis of acute angle.* [*An equidistantial of a straight has its chords between it and the straight.*]

Fig. 42.

Let the straight CD join equal perpendiculars AC, BD standing upon any straight AB. Then AB, CD being bisected in the points M and H (Fig. 42.), MH is joined perpendicular (by Proposition II) to each. Again in this hypothesis the angles at the join CD are supposed acute. Therefore in the quadrilateral $AMHC$ (by Corollary I after Proposition III) MH will be less than AC. Hence now, if in MH produced MK be taken equal to AC, the points C, K, D pertain to the curve here investigated. Then the angles at the join CK will be themselves acute (by Proposition VII).

Therefore the join LX, which bisects, and therefore (by Proposition II), is at right angles to AM, CK, will be likewise (by Corollary I after Proposition III) less than AC. Wherefore, if in LX produced we assume LF equal to AC or MK, the point F also will pertain to this curve. Further, joining CF, and

FK we find likewise two other points pertaining to the same curve. And so on for ever.

But what I say for finding points between the points *C* and *K*, the same also holds good uniformly for finding points between the points *K* and *D*. Obviously the curve *CKD*, arising from the hypothesis of acute angle, is the line joining the extremities of all equal perpendiculars erected upon the same base toward the same part, which assuredly can come under the name ordinates. It is, I add, a line of such sort, that on account of the hypothesis of acute angle, from which it arises, it always is concave toward the parts of the opposite base *AB*. Quod quidem hoc loco declarandum, ac demonstrandum a nobis erat.

PROPOSITION XXXV. *If from any point L of the base AB the ordinate LF is drawn to this curve CKD: I say the straight NFX perpendicular to LF must on both sides fall wholly toward the convex parts of this curve, and therefore it will be tangent to this curve.*

Proof. For if possible, let a certain point *X* (Fig. 43.) of *NFX* fall within the cavity of this curve. Let fall from the point *X* to the base *AB* the perpendicular *XP*, which prolonged through *X* meets the curve in a certain point *R*. Now thus. In the quadrilateral *LFXP* the angle at the point *X* will be neither right nor obtuse: else (Proposition V and Proposition VI) would be destroyed the present hypothesis of acute angle.

Therefore the aforesaid angle will be acute. Wherefore (from Corollary I after Proposition III) *PX* and so much more *PR* will be greater than *LF*. But this is absurd (from the preceding) against the nature of this curve.

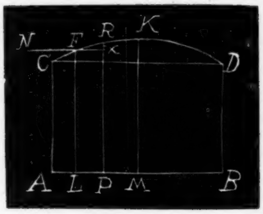


Fig. 43.

So *NF* produced must fall wholly toward the convex parts, and so it will be tangent to this curve. Quod erat demonstrandum.

[To be Continued.]

SUMMATION OF SERIES.

By G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

$$\frac{\sin h\theta}{\theta}=\left(1+\frac{\theta^2}{\pi^2}\right)\left(1+\frac{\theta^2}{2^2\pi^2}\right)\left(1+\frac{\theta^2}{3^2\pi^2}\right)\left(1+\frac{\theta^2}{4^2\pi^2}\right)\ldots\ldots$$

$$=1+\frac{\theta^2}{3\,!}+\frac{\theta^4}{5\,!}+\frac{\theta^6}{7\,!}+\frac{\theta^8}{9\,!}+\ldots\ldots(A).$$

$$\therefore \log\left(1+\frac{\theta^2}{\pi^2}\right)+\log\left(1+\frac{\theta^2}{2^2\pi^2}\right)+\log\left(1+\frac{\theta^2}{3^2\pi^2}\right)+\log\left(1+\frac{\theta^2}{4^2\pi^2}\right)\ldots\ldots$$

$$=\log\left[1+\left(\frac{\theta^2}{3\,!}+\frac{\theta^4}{5\,!}+\frac{\theta^6}{7\,!}+\frac{\theta^8}{9\,!}+\ldots\ldots\right)\right].$$

$$\therefore \frac{\theta^2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots\ldots\right)$$

$$-\frac{\theta^4}{2\pi^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots\ldots\right)$$

$$+\frac{\theta^6}{3\pi^6}\left(\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\ldots\ldots\right)$$

$$-\frac{\theta^8}{4\pi^8}\left(\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\ldots\ldots\right)\ldots\ldots$$

$$=\frac{\theta^2}{6}-\frac{\theta^4}{180}+\frac{\theta^6}{2835}-\frac{\theta^8}{37800}+\ldots\ldots$$

$$\therefore \frac{1}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots\ldots\right)=\tfrac{1}{6},$$

$$\text{and}\ \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots\ldots=\frac{\pi^2}{6}\ \ldots\ldots\ldots\ (1),$$

$$-\frac{1}{2\,\pi^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots\ldots\right)=-\frac{1}{180}$$

$$\text{and}\ \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots\ldots=\frac{\pi^4}{90}\ldots\ldots\ldots\ldots(2),$$

$$\frac{1}{3\pi^6}\left(\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\ldots\ldots\right)=\frac{1}{2835}$$

$$\text{and}\ \frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\ldots\ldots=\frac{\pi^6}{945}\ldots\ldots\ldots(3),$$

$$-\frac{1}{4\pi^8}\left(\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\ldots\ldots\right)=-\frac{1}{37800}$$

$$\text{and}\ \frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\ldots\ldots=\frac{\pi^8}{9450}\ldots\ldots\ldots(4),$$

$$\cosh\theta=\left(1+\frac{4\theta^2}{\pi^2}\right)\left(1+\frac{4\theta^2}{3^2\pi^2}\right)\left(1+\frac{4\theta^2}{5^2\pi^2}\right)\left(1+\frac{4\theta^2}{7^2\pi^2}\right)\ldots\ldots$$

$$=1+\frac{\theta^2}{2!}+\frac{\theta^4}{4!}+\frac{\theta^6}{6!}+\frac{\theta^8}{8!}+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (B).$$

$$\therefore \log\left(1+\frac{4\theta^2}{\pi^2}\right)+\log\left(1+\frac{4\theta^2}{3^2\pi^2}\right)+\log\left(1+\frac{4\theta^2}{5^2\pi^2}\right)+\ \ldots\ldots$$

$$=\log\left[1+\left(\frac{\theta^2}{2!}+\frac{\theta^4}{4!}+\frac{\theta^6}{6!}+\frac{\theta^8}{8!}+\ \ldots\ldots\right)\right].$$

$$\therefore \frac{4\theta^2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ \ldots\ldots\right)-\frac{16\theta^4}{2\pi^4}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ \ldots\ldots\right)$$

$$+\frac{64\theta^6}{3\pi^6}\left(\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\ \ldots\ldots\right)-\frac{256\theta^8}{4\pi^8}\left(\frac{1}{1^8}+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\ \ldots\ldots\right)$$

$$\ldots\ldots=\frac{\theta^2}{2}-\frac{\theta^4}{12}+\frac{\theta^6}{45}-\frac{17\theta^8}{2520}+\ \ldots\ldots$$

$$\therefore \frac{4}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ \ldots\ldots\right)=\tfrac{1}{2}$$

$$\text{and } \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ \ldots\ldots=\frac{\pi^2}{8}\ \ldots\ldots\ldots\ldots\ (5),$$

$$-\frac{16}{2\pi^4}\left(\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ \ldots\ldots\right)=-\tfrac{1}{12}$$

$$\text{and } \frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ \ldots\ldots=\frac{\pi^4}{96}\ \ldots\ldots\ldots\ (6),$$

$$\frac{64}{3\pi^6}\left(\frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\ \ldots\ldots\right)=\tfrac{1}{45}$$

$$\text{and } \frac{1}{1^6}+\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}+\ \ldots\ldots=\frac{\pi^6}{960}\ \ldots\ldots\ldots\ (7),$$

$$-\frac{256}{4\pi^8}\left(\frac{1}{1^8}+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\ \ldots\ldots\right)$$

$$\text{and } \frac{1}{1^8}+\frac{1}{3^8}+\frac{1}{5^8}+\frac{1}{7^8}+\ \ldots\ldots=\frac{17\pi^8}{161280}\ \ldots\ldots\ (8).$$

(1)−(5) gives $\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ \ldots\ldots=\frac{\pi^2}{24}\ \ldots\ldots\ldots\ldots\ (9),$

(2)−(6) gives $\frac{1}{2^4}+\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\ \ldots\ldots=\frac{\pi^4}{1440}\ \ldots\ldots\ldots\ldots\ (10),$

(3)−(7) gives $\frac{1}{2^6}+\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\ \ldots\ldots=\frac{\pi^6}{60480}\ \ldots\ldots\ldots\ (11),$

(4)−(8) gives $\frac{1}{2^8}+\frac{1}{4^8}+\frac{1}{6^8}+\frac{1}{8^8}+\ldots\ldots=\frac{\pi^8}{2419200}$ .... (12),

(5)−(9) gives $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\ldots\ldots=\frac{\pi^2}{12}$ .... (13),

(6)−(10) gives $\frac{1}{1^4}-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\frac{1}{6^4}+\ldots\ldots=\frac{7\pi^4}{720}$ .... (14).

(7)−(11) gives $\frac{1}{1^6}-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\ldots\ldots=\frac{31\pi^6}{30240}$ .... (15).

(8)−(12) gives $\frac{1}{1^8}-\frac{1}{2^8}+\frac{1}{3^8}-\frac{1}{4^8}+\frac{1}{5^8}-\frac{1}{6^8}+\ldots\ldots=\frac{11\pi^8}{172800}$ ...... (16),

½ of (1) gives $\frac{1}{1.2}+\frac{1}{2.4}+\frac{1}{3.6}+\frac{1}{4.8}+\ldots\ldots=\frac{\pi^2}{12}=(13)$ ...... (17),

½ of (5) gives $\frac{1}{1.2}+\frac{1}{3.6}+\frac{1}{5.10}+\frac{1}{7.14}+\ldots\ldots=\frac{\pi^2}{16}$ .... (18),

⅓ of (1) gives $\frac{1}{1.3}+\frac{1}{2.6}+\frac{1}{3.9}+\frac{1}{4.12}+\ldots\ldots=\frac{\pi^2}{18}$ .... (19),

⅓ of (5) gives $\frac{1}{1.3}+\frac{1}{3.9}+\frac{1}{5.15}+\frac{1}{7.21}+\ldots\ldots=\frac{\pi^2}{24}=(9)$ .... (20),

¼ of (1) gives $\frac{1}{1.4}+\frac{1}{2.8}+\frac{1}{3.12}+\frac{1}{4.16}+\ldots\ldots=\frac{\pi^2}{24}=(9)=(20)$ .. (21).

¼ of (5) gives $\frac{1}{1.4}+\frac{1}{3.12}+\frac{1}{5.20}+\frac{1}{7.28}+\ldots\ldots=\frac{\pi^2}{32}$ .... (22).

(17)−(18) gives $\frac{1}{2.4}+\frac{1}{4.8}+\frac{1}{6.12}+\frac{1}{8.16}+\ldots\ldots=\frac{\pi^2}{48}$ .... (23),

(19)−(20) gives $\frac{1}{2.6}+\frac{1}{4.12}+\frac{1}{6.18}+\frac{1}{8.24}+\ldots\ldots=\frac{\pi^2}{72}$ .... (24),

(21)−(22) gives $\frac{1}{2.8}+\frac{1}{4.16}+\frac{1}{6.24}+\frac{1}{8.32}+\ldots\ldots=\frac{\pi^2}{96}$ .... (25),

(1)−(2) gives $\frac{1}{1^2}-\frac{1}{2^4}+\frac{1}{3^2}-\frac{1}{3^4}+\frac{1}{4^2}-\frac{1}{4^4}+\frac{1}{5^2}-\frac{1}{5^4}+\ldots\ldots=\frac{\pi^2}{6}\left(1-\frac{\pi^2}{15}\right)$

$\therefore\frac{3}{2^4}+\frac{8}{3^4}+\frac{15}{4^4}+\frac{24}{5^4}+\ldots\ldots=\frac{\pi^2}{6}\left(1-\frac{\pi^2}{15}\right)$ .... (26).

(2)−(3) gives $\frac{3}{2^6}+\frac{8}{3^6}+\frac{15}{4^6}+\frac{24}{5^6}+\ \ldots\ldots=\frac{\pi^4}{45}\left(\frac{1}{2}-\frac{\pi^2}{21}\right)$ ....(27),

(3)−(4) gives $\frac{3}{2^8}+\frac{8}{3^8}+\frac{15}{4^8}+\frac{24}{5^8}+\ \ldots\ldots=\frac{\pi^6}{945}\left(1-\frac{\pi^2}{10}\right)$ ....(28),

(5)−(6) gives $\frac{8}{3^4}+\frac{24}{5^4}+\frac{48}{7^4}+\frac{80}{9^4}+\ \ldots\ldots=\frac{\pi^2}{8}\left(1-\frac{\pi^2}{12}\right)$ ....(29),

(6)−(7) gives $\frac{8}{3^6}+\frac{24}{5^6}+\frac{48}{7^6}+\frac{80}{9^6}+\ \ldots\ldots=\frac{\pi^4}{96}\left(1-\frac{\pi^2}{10}\right)$ ....(30),

(7)−(8) gives $\frac{8}{3^8}+\frac{24}{5^8}+\frac{48}{7^8}+\frac{80}{9^8}+\ \ldots\ldots=\frac{\pi^6}{960}\left(1-\frac{17\pi^2}{168}\right)$ ....(31),

(9)−(10) gives so also (26)−(29)

$$\frac{3}{2^4}+\frac{15}{4^4}+\frac{35}{6^4}+\frac{63}{8^4}+\ \ldots\ldots=\frac{\pi^4}{24}\left(1-\frac{\pi^2}{60}\right) \quad\ldots\ldots(32),$$

(10)−(11) gives so also (27)−(30)

$$\frac{3}{2^6}+\frac{15}{4^6}+\frac{35}{6^6}+\frac{63}{8^4}+\ \ldots\ldots=\frac{\pi^4}{1440}\left(1-\frac{\pi^2}{42}\right) \quad\ldots\ldots(33),$$

(11)−(12) gives so also (28)−(31)

$$\frac{3}{2^8}+\frac{15}{4^8}+\frac{35}{6^8}+\frac{63}{8^8}+\ \ldots\ldots=\frac{\pi^6}{60480}\left(1-\frac{\pi^2}{40}\right) \quad\ldots\ldots(34),$$

$\frac{1}{8}$ of (29) gives $\frac{1}{3^4}+\frac{3}{5^4}+\frac{6}{7^4}+\frac{10}{9^4}+\ \ldots\ldots=\frac{\pi^2}{64}\left(1-\frac{\pi^2}{12}\right)$ ....(35),

$\frac{1}{8}$ of (30) gives $\frac{1}{3^6}+\frac{3}{5^6}+\frac{6}{7^6}+\frac{10}{9^6}+\ \ldots\ldots=\frac{\pi^2}{768}\left(1-\frac{\pi^2}{10}\right)$ ....(36),

$\frac{1}{8}$ of (31) gives $\frac{1}{3^8}+\frac{3}{5^8}+\frac{6}{7^8}+\frac{10}{9^8}+\ \ldots\ldots=\frac{\pi^6}{7680}\left(1-\frac{17\pi^2}{168}\right)$ .....(37).

Squaring (1) gives $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ \ldots$

$$+\frac{2}{1^2.2^2}+\frac{2}{1^2.3^2}+\frac{2}{2^2.3^2}+\ \ldots\ldots=\frac{\pi^4}{36}.$$

$$\therefore \frac{1}{1^2.2^2}+\frac{1}{1^2.3^2}+\frac{1}{1^2.4^2}+\frac{1}{2^2.3^2}+\frac{1}{2^2.4^2}+\frac{1}{3^2.4^2}+\ \ldots\ldots$$

$$=\tfrac{1}{2}\left(\frac{\pi^4}{36}-\frac{\pi^4}{90}\right)=\frac{\pi^4}{120}=\frac{\pi^4}{5\,!}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (38).$$

Squaring (2) gives with (4)

$$\frac{1}{1^4.2^4}+\frac{1}{1^4.3^4}+\frac{1}{1^4.4^4}+\frac{1}{2^4.3^4}+\frac{1}{2^4.4^4}+\frac{1}{3^4.4^4}+\ldots.$$

$$=\tfrac{1}{2}\left(\frac{\pi^8}{8100}-\frac{\pi^8}{9450}\right)=\frac{\pi^8}{113400}\ \ldots\ldots\ldots\ldots\ldots (39).$$

Squaring (5) gives with (6)

$$\frac{1}{1^2.3^2}+\frac{1}{1^2.5^2}+\frac{1}{1^2.7^2}+\frac{1}{3^2.5^2}+\frac{1}{3^2.7^2}+\frac{1}{5^2.7^2}+\ldots.$$

$$=\tfrac{1}{2}\left(\frac{\pi^4}{64}-\frac{\pi^4}{96}\right)=\frac{\pi^4}{384}=\frac{\pi^2}{4^2(4!)}\ \ldots\ldots\ldots\ldots\ldots (40).$$

Squaring (6) gives with (8)

$$\frac{1}{1^4.3^4}+\frac{1}{1^4.5^4}+\frac{1}{1^4.7^4}+\frac{1}{3^4.5^4}+\frac{1}{3^4.7^4}+\frac{1}{5^4.7^4}+\ldots.$$

$$=\tfrac{1}{2}\left(\frac{\pi^2}{9216}-\frac{17\pi^8}{161280}\right)=\frac{\pi^8}{645120}\ \ldots\ldots\ldots\ldots\ldots (41).$$

Squaring (9) gives with (10)

$$\frac{1}{2^2.4^2}+\frac{1}{2^2.6^2}+\frac{1}{2^2.8^2}+\frac{1}{4^2.6^2}+\frac{1}{4^2.8^2}+\frac{1}{6^2.8^8}+\ \ldots\ldots$$

$$=\tfrac{1}{2}\left(\frac{\pi^4}{576}-\frac{\pi^2}{1440}\right)=\frac{\pi^4}{1920}\ \ldots\ldots\ldots\ldots\ldots (42).$$

Squaring (10) gives with (12)

$$\frac{1}{2^4.4^4}+\frac{1}{2^4.6^4}+\frac{1}{2^4.8^4}+\frac{1}{4^4.6^4}+\frac{1}{4^4.8^4}+\frac{1}{6^4.8^4}+\ \ldots\ldots$$

$$=\tfrac{1}{2}\left(\frac{\pi^8}{2073600}-\frac{\pi^8}{241920}\right)=\frac{\pi^8}{29030400}\ \ldots\ldots\ldots (43).$$

$$1/\left[\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ \ldots\right]=\left(1+\frac{1}{2^2}+\frac{1}{2^4}+\frac{1}{2^6}+\ldots.\right)$$

$$\left(1+\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\ldots.\right)\left(1+\frac{1}{5^2}+\frac{1}{5^4}+\frac{1}{5^6}+\ldots.\right)\ \ldots\ .$$

$$=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots=\tfrac{1}{6}\pi^2 \quad ..(44),$$

$$1/\left[\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\ldots\ldots\right]$$

$$=\left(1+\frac{1}{2^4}+\frac{1}{2^8}+\frac{1}{2^{12}}+\ldots\right)\left(1+\frac{1}{3^4}+\frac{1}{3^8}+\frac{1}{3^{12}}+\ldots\right)$$

$$\left(1+\frac{1}{5^4}+\frac{1}{5^8}+\frac{1}{5^{12}}+\ldots\right)\ldots$$

$$=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots\ldots=\frac{\pi^4}{90} \quad \ldots\ldots\ldots (45).$$

Similarly,

$$1/\left[\left(1-\frac{1}{2^6}\right)\left(1-\frac{1}{3^6}\right)\left(1-\frac{1}{5^6}\right)\left(1-\frac{1}{7^6}\right)\ldots\ldots\right]=\frac{\pi^6}{945}\ldots\ldots\ldots (46).$$

$$1/\left[\left(1-\frac{1}{2^8}\right)\left(1-\frac{1}{3^8}\right)\left(1-\frac{1}{5^8}\right)\left(1-\frac{1}{7^8}\right)\ldots\ldots\right]=\frac{\pi^8}{9450}\ldots\ldots\ldots (47).$$

Hence,

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ldots\ldots=\frac{6}{\pi^2}\ldots\ldots\ldots (48).$$

$$\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\ldots\ldots=\frac{90}{\pi^4}\ldots\ldots\ldots (49).$$

$$\left(1-\frac{1}{2^6}\right)\left(1-\frac{1}{3^6}\right)\left(1-\frac{1}{5^6}\right)\left(1-\frac{1}{7^6}\right)\ldots\ldots=\frac{945}{\pi^6}\ldots\ldots\ldots (50).$$

$$\left(1-\frac{1}{2^8}\right)\left(1-\frac{1}{3^8}\right)\left(1-\frac{1}{5^8}\right)\left(1-\frac{1}{7^8}\right)\ldots\ldots=\frac{9450}{\pi^8}\ldots\ldots\ldots (51).$$

$$\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\left(1+\frac{1}{7^2}\right)\ldots\ldots$$

$$=\frac{\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\ldots\ldots}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ldots\ldots}=\frac{\frac{90}{\pi^4}}{\frac{6}{\pi^2}}=\frac{15}{\pi^2}\ldots\ldots\ldots (52).$$

$$\left(1+\frac{1}{2^4}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{5^4}\right)\left(1+\frac{1}{7^4}\right)\ldots$$

$$=\frac{\left(1-\frac{1}{2^8}\right)\left(1-\frac{1}{3^8}\right)\left(1-\frac{1}{5^8}\right)\left(1-\frac{1}{7^8}\right)\cdots\cdots}{\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\cdots\cdots}=\frac{\frac{9450}{\pi^8}}{\frac{90}{\pi^4}}=\frac{105}{\pi^4}\quad\cdots\cdots\cdots(53).$$

$$\left(1+\frac{1}{2^2}+\frac{1}{2^4}\right)\left(1+\frac{1}{3^2}+\frac{1}{3^4}\right)\left(1+\frac{1}{5^2}+\frac{1}{5^4}\right)\left(1+\frac{1}{7^2}+\frac{1}{7^4}\right)\cdots\cdots$$

$$=\frac{\left(1-\frac{1}{2^6}\right)\left(1-\frac{1}{3^6}\right)\left(1-\frac{1}{5^6}\right)\left(1-\frac{1}{7^6}\right)\cdots\cdots}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\cdots\cdots}=\frac{\frac{945}{\pi^6}}{\frac{6}{\pi^2}}=\frac{315}{2\,\pi^4}\quad\cdots\cdots\cdots(54).$$

From (A) by inspection,

$$\frac{1}{1^2.2^2.3^2}+\frac{1}{1^2.2^2.4^2}+\frac{1}{1^2.3^2.4^2}+\frac{1}{2^2.3^2.4^2}+\cdots\cdots=\frac{\pi^6}{7\,!}\quad\cdots\cdots(55).$$

$$\frac{1}{1^2.2^2.3^2.4^2}+\frac{1}{1^2.2^2.3^2.5^2}+\frac{1}{1^2.3^2.4^2.5^2}+\cdots\cdots=\frac{\pi^8}{9\,!}\quad\cdots\cdots(56).$$

Generally $\frac{1}{1^2.2^2\ldots(n-1)^2}+\frac{1}{1^2.3^2\ldots n^2}+\frac{1}{1^2.2^2.4^2\ldots n^2}+\ldots=\frac{\pi^{2n}}{(2n+1)!}\quad\cdots(57).$

When $\theta=\pi$, $\frac{1}{\pi}\sinh\pi=\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{4^2}\right)\left(1+\frac{1}{5^2}\right)\quad\cdots(58).$

$$=1+\frac{\pi^2}{3\,!}+\frac{\pi^4}{5\,!}+\frac{\pi^6}{7\,!}+\frac{\pi^8}{9\,!}+\ldots(59)=1+(1)+(38)+(55)+(56)+\ldots(60).$$

From (B) by inspection,

$$\frac{1}{1^2.3^2.5^2}+\frac{1}{1^2.3^2.7^2}+\frac{1}{3^2.5^2.7^2}+\cdots\cdots=\frac{\pi^6}{4^3(6\,!)}\quad\cdots\cdots(61).$$

$$\frac{1}{1^2.3^2.5^2.7^2}+\frac{1}{1^2.3^2.5^2.9^2}+\frac{1}{3^2.5^2.7^2.9^2}+\cdots\cdots=\frac{\pi^8}{4^4(8\,!)}\quad\cdots\cdots(62).$$

Generally $\frac{1}{1^2.3^2\text{ to }n\text{ factors}}+\frac{1}{1^2.5^2\text{ to }n\text{ factors}}+\cdots\cdots=\frac{\pi^{2n}}{4^n(2n\,!)}\quad\cdots\cdots(63).$

When $\theta=\pi$, $\cosh\pi=\left(1+\frac{4}{1^2}\right)\left(1+\frac{4}{3^2}\right)\left(1+\frac{4}{5^2}\right)\left(1+\frac{4}{7^2}\right)\cdots\cdots\cdots(64)$

$$=1+\frac{\pi^2}{2\,!}+\frac{\pi^4}{4\,!}+\frac{\pi^6}{6\,!}+\frac{\pi^8}{8\,!}+.(65)=1+4(5)+4^2(40)+4^3(61)+4^4(62)+.(66).$$

$$2\cosh\pi=e^{\pi}+e^{-\pi}=2(1+2^2)[1+(\tfrac{2}{3})^2][1+(\tfrac{2}{5})^2][1+(\tfrac{2}{7})^2]\quad\cdots\cdots(67).$$

$$2\sinh\pi=e^{\pi}-e^{-\pi}=4\pi[1+(\tfrac{1}{2})^2][1+(\tfrac{1}{3})^2][1+(\tfrac{1}{4})^2][1+(\tfrac{1}{5})^2]\quad\cdots\cdots(68).$$

The number of series summed above is deemed sufficient for illustration, although many more could be summed from the above relations.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

90. **Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

Find the greatest number of inch balls that can be placed in a box 10 inches square and 5 inches deep.

Solution by MARTIN SPINX, Wilmington, Ohio, and the PROPOSER.

Using as a base a side of the box, we can place on this base, by square arrangement 50 balls. But by placing 5 balls in the first row, 4 in the second, and 5 in the third, and so on, we can place 50 balls in the first layer, there being 6 rows of 5 balls, and 5 rows of 4 balls. These eleven rows will leave .339 inches between the eleventh row and the end of the box. By placing the second layer of balls in the trihedral spaces of the first layer, the two layers will occupy a space 1 inch $+\sqrt{\{1^2-[\frac{2}{3}(\frac{1}{2}\sqrt{3})]^2\}}$ inches, $=1.8165$ inches, high.

Since the centers of the first row in the second layer are .289 inches in advance of the centers of the first row in the first layer, it follows that eleven rows can be put in the second layer, there being .339 inches—.289 inches, or .05 inches more room than is needed. But the second layer contains 49 balls,—the six odd rows containing 4 each and the five even rows 5 each. In this we can place in the box twelve layers,—50 balls in each of the odd numbered layers, and 49 in each of the even numbered layers. This makes a total of 594 balls.

Also solved, with different results, by *G. B. M. ZERR, CHAS. C. CROSS,* and *FREMONT CRANE.*

90. **Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.**

A owes $6000 which is drawing 6% interest. He wishes to pay off the debt in six equal annual payments, the first to be due in one year. The whole portion of the claim unpaid at the end of each year to be accounted as principal, and to draw interest to the time of the next payment. Required the amount of each payment, so the six annual payments will discharge the obligation, interest and all.

I. **Proposed by P. S. BERG, Superintendent of Schools, Larimore, N. D.; J. A. MOORE, Professor of Mathematics, Millsaps College, Millsaps, Miss.; M. E. GRABER, Mt. Eaton, O.; MARTIN SPINX, Wilmington, O., M. A. GRUBER, A. M., War Department, Washington, D. C.**

Let x=annual payment.

Put a=debt, r=rate of interest, and n=number of annual payments.

We then have the general formula,

$$\frac{(100+r)\left\{\begin{array}{c}\dfrac{\ldots\ldots(100+r)\left(\dfrac{(100+r)a}{100}-x\right)}{100}-x \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \text{to } n\text{th payment}\end{array}\right\}}{100}-x=0,$$

which reduces to

$$a(100+r)^n=x[100(100+r)^{n-1}+100^2(100+r)^{n-2}+\ \ldots\ldots+100^{n-1}(100+r)+100^n].$$

Now, substituting 6 for r, 6 for n, and 6000 for a, we obtain

$$6000\times1.06^6=x(1.06^5+1.06^4+1.06^3+1.06^2+1.06+1).$$

$\therefore\ x=8511.114673536\div6.9753185376=1220.1757+.$

II. Solution by F. R. HONEY, Ph. B., New Haven, Conn.

Let x=amount of each annual payment.

$\therefore\ 6000+.06\times6000-x=6360-x$=amount left after first payment.

And $6360-x+.06(6360-x)-x=6741.6-2.06x$=amount left after second payment.

Similarly, $7146.096-3.1836x$=amount left after third payment.

Similarly, $7574.86176-4.374616x$=amount left after fourth payment.

Similarly, $8029.3534656-5.63709296x$=amount left after fifth payment.

Similarly, $8511.114673536-6.9753185376x$=amount after sixth payment.

Then $8511.114673536-6.9753185376x=0$.

$\therefore\ 6.9753185376x=8511.114673536$.

$\therefore\ x=\$1220.176$=annual payment.

III. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.; W. H. DRANE, A. M., Professor of Mathematics, Jefferson Military College, Washington, Miss.

The portions of the principal paid each year are as

1, 1/1.06, $1/(1.06)^2$, $1/(1.06)^3$, $1/(1.06)^4$, $1/(1.06)^5$; or as $(1.06)^5$, $(1.06)^4$, $(1.06)^3$, $(1.06)^2$, 1.06, 1 ; or as 1.338226, 1.262477, 1.191016, 1.1236, 1.06, 1.

$1.338226+1.262477+1.191016+1.1236+1.06+1\doteq6.976319$.

$(1/6.975319)\times\$6000=\$860.1757$.

$\$860.1756+\$6000\times.06=\$860.1758+\$360=\$1220.1757$, first payment.

$\$6000-\$860.1757=\$5139.8243$, amount still unpaid.

$(1.06/6.975319)\times\$6000=\$911.7862$.

$\$911.7862+\$5139.8243\times.06=\$911.7862+\$308.3895=\$1220.1757$, second payment.

$\$5139.8243-\$911.7862=\$4228.0381$, amount still unpaid.

$[(1.06)^2/6.975319]\times\$6000=\$966.4934$.

$966.4934+$4228.0381 × .06=$966.4934+$253.6823=$1220.1757, third payment.

$4228.0381−$966.4934=$3261.5447, amount still unpaid.

$[(1.06)^3/6.975319]\times\$6000=\$1024.4830$.

$1024.4830+$3261.5447 × .06=$1024.4830+$195.6927=$1220.1757, 4th payment.

$3261.5447−$1024.4830=$2237.0617, amount still unpaid.

$[(1.06)^4/6.975319]\times\$6000=\$1085.9520$.

$1085.9520+$2237.0617 × .06=$1085.9520+$134.2237=$1220.1757, fifth payment.

$2237.0617−$1085.9520=$1151.1097, amount still unpaid.

$[(1.06)^5/6.975319]\times\$6000=\$1151.1091$.

$1151.1091+$1151.1097 × .06=$1151.1091+$69.0666=$1220.1757, sixth payment.

$1151.1097−$1151.1091=$0.0006, unpaid still.

The above is all the work necessary for determining each equal payment, and at the same time working out the problem in full.

A short method by algebra is, $p=\dfrac{6000\times .06(1.06)^6}{(1.06)^6-1}=\$1220.176$.

91. **Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics in Curry University, Pittsburg, Pa.**

$1000.00. Cleveland, Ohio, May 26, 1893.

Two years after date I promise to pay John Davis, or order, one thousand dollars, for value received, interest six per cent. payable annually. J. M. Lewis.

Indorsements: December 14, 1895, $560.56; May 11, 1896, $10.02; June 14, 1897, $545.06.

Find, by the United States' Rule, the amount due August 2, 1897.

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.; WALTER HUGH DRANE, A. M., Jefferson Military College, Washington, Miss.; and MARTIN SPINX, Wilmington, Ohio.

Principal,		$1000.00
Interest to December 14, 1895, 2 years, 6 months, 18 days,		153.00
Amount,		$1153.00
First payment December 14, 1895,		560.56
New principal,		$ 592.44
Interest to June 14, 1897, 1 year, 6 months,		53.3196
Amount,		$ 645.7596
Second payment, May 11, 1896,	10.02	
Third payment, June 14, 1897,	545.06	555.08
New principal,		$ 90.6796
Interest to August 1, 1897, 1 month, 18 days,		.7254
Amount or balance due,		$91.4050

Daniel G. Dorrance gets as a result $100.80. He computed interest on the annual payments. Mr. Drane, in a second solution does the same, but he gets a result of $100.138.

REMARK. Mr. Drummond raises the question as to whether Mr. J. F. Travis's solution of problem 87, Arithmetic, is strictly an arithmetical solution. To my mind it is strictly an algebraical solution. A pure arithmetical solution of a problem would involve only the operations of addition, subtraction, multiplication, division, involution, and evolution, without the use of equations. A solution in which the result sought is represented by some character, and then this character operated upon until certain conditions of the problem are fulfilled, which conditions are then stated in the form of an equation from which the numerical value of the character is to be determined, is an algebraic solution. It is immaterial what sort of a character is used, whether it be $(\frac{2}{2})$, $\frac{3}{3}$, x, ϕ, or any other character. However, the solution referred to is a very good one, and by the use of such solutions students in arithmetic are given, unconsciously to themselves, a most excellent preparation for the study of algebra. The mathematician is often called upon to solve problems in a certain way. When a problem is proposed and the restriction put upon it, viz., that it be solved by arithmetic, or algebra, or geometry, the problem often becomes impossible. From such unfortunate restrictions, has arisen the idea of the insolvability of the three famous problems of geometry, viz., the Trisection of an Angle, the Duplication of the Cube, and the Quadrature of the Circle. These problems are each easily solved if the solutions are not restricted to the use of the straight edge and compass only. But with these restrictions they are absolutely unsolvable.

There are many problems whose solutions cannot be effected when restricted in the way previously mentioned, but those referred to above are the only ones that have become famous.

ALGEBRA.

81. II. Solution by C. W. M. BLACK, A. M., Professor of Mathematics, Wesleyan Academy, Wilbraham, Mass.

[See problem and solution I, in April number, page 105.] The proposition cannot be proved unless r is integral and positive, as can be shown by substitution of numerical values.

Consider the only two fractions in whose denominators any factor as (a_1-a_2) appears, putting them in the form

$$(a_1{}^r)/[(a_1-a_2)(a_1-a_3)(a_1-a_4)\;.....(a_1-a_n)]-(a_2{}^r)$$
$$/[(a_1-a_2)(a_2-a_3)(a_2-a_4)\;.....(a_2-a_n)]=(a_1{}^r)$$
$$/[(a_1-a_2)(a_1{}^{n-2}P_1a_1{}^{n-3}+P_2a_1{}^{n-4}-\;......\pm P_{n-2})]$$
$$-(a_2{}^r)/[(a_1-a_2)(a_2{}^{n-2}P_1a_2{}^{n-3}+P_2a^{n-4}-\;.....\pm P_{n-2})],$$

where P_k=the sum of the products of a_3, a_4, ...a_n taken k at a time.

Combining, we have

$$[a_1^r(a_2^{n-2}-P_1a_2^{n-3}+P_2a_2^{n-4}-\ldots\ldots\pm P_{n-2})$$
$$-a_2^r(a_1^{n-2}-P_1a_1^{n-3}+P_2a_1^{n-4}-\ldots\ldots P_{n-2})]$$
$$/[(a_1-a_2)(a_1^{n-2}-P_1a^{n-3}+P_2a_1^{n-4}-\ldots\ldots\pm P_{n-2})$$
$$(a_2^{n-2}-P_1a_2^{n-3}+P_2a_2^{n-4}-\ldots\ldots\pm P_{n-2})]\ \ldots\ldots\ldots\ldots(1).$$

Put the numerator of (1) in the form

$$(a_1^ra_2^{n-2}-a^ra_1^{n-2})-P_1(a_1^ra_2^{n-3}-a_2^ra_1^{n-3})+\ldots\ldots\ \pm P_{n-2}(a_1^r-a_2^r)$$
$$=a_1^{n-2}a_2^{n-2}(a_1^{\,r-n+2}-a_2^{\,r-n+2})-P_1a_1^{n-3}a_2^{n-3}(a_1^{\,r-n+3}-a_2^{\,r-n+3})$$
$$+\ldots\ldots\pm P_{n-2}(a_1^r-a_2^r)\ldots\ldots(2).$$

If n is not greater than $r+2$ each group of (2) and consequently the whole expression is divisible by (a_1-a_2). If $n>r+2$, let $n=r+s$; then change (2) to the form

$$a_1^ra_2^r[(a_2^{\,n-2-r}-a_1^{\,n-2-r})-P_1(a_2^{\,n-3-r}-a_1^{\,n-3-r})+\ldots\ldots$$
$$\pm P_{s-2}(a_2^{\,n-s-r}-a_1^{\,n-s-r})]\mp[P_{s-1}a_1^{\,n-s-1}a_2^{\,n-s-1}(a_1^{\,r-n+s+1}-a_2^{\,r-n+s+1})$$
$$-P_sa_1^{n-s-2}a_2^{n-s-2}(a_1^{\,r-n+s+2}-a_2^{\,r-n+s+2})$$
$$+\ \ldots\ldots\pm P_{n-2}(a_1^r-a_2^r)]\ldots\ldots(3),$$

the term in the second group of (3) being the same as the corresponding term of (2). Each group in (3) is also divisible by (a_1-a_2). Accordingly in all cases (1) can be reduced to a form in which (a_1-a_2) is not a factor of the denominator, and as the two fractions forming (1) are the only ones that contain (a_1-a_2) in their denominators, the original expression need not contain (a_1-a_2) in its denominator; that is, (a_1-a_2) will divide into the numerator formed by adding the fractions as they stand.

In like manner we prove that any other factor (a_2-a_3), etc., will divide into the numerator, or the numerator will be divisible by the entire lowest common denominator.

Now if $r<n-1$ each fraction, and consequently the sum of all, will have a numerator of lower degree in a_1, a_2, a_3, etc., than the denominator. But as the numerator is divisible by the denominator, this is possible only when the numerator equals zero.

If $r=n-1$, numerator and denominator will have same degree, and the quotient can be only a numerical factor. Now in the numerator a_1^r has for its coefficient the product of all factors not containing a_1, which same coefficient it has in the expansion of the denominator. Therefore the quotient must equal 1.

If $r=n$ the numerator is of a degree 1 higher than the denominator and the quotient must be of the first degree. In the numerator the coefficient of a_1^r is the same as the coefficient of $a_1^{n-1}(=a_1^{r-1})$ in denominator, these being the highest powers of a_1 in each. Then one term of the quotient must be a_1. In like manner we show that a_2, a_3, ...a_n must all be true of quotient, and as the expression is symmetrical with respect to these, the value must be

$$a_1+a_2+a_3+\dots\dots+a_n.$$

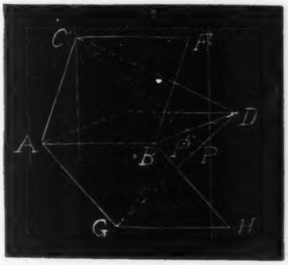
GEOMETRY.

87. **Proposed by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military Academy, Washington, Miss.**

Given any two straight lines in space, AB, CD, which do not intersect. So construct upon one of the lines as base, a triangle, having its vertex in the other line, such that its perimeter shall be a minimum.

I. **Solution by the PROPOSER.**

Let AB and CD be the given straight lines. Pass planes through the line AB and the points C and D. In the plane of $ABDE$, inclined the same way and making the same angle with the line AB as $ABFC$, construct a parallelogram equal to $CABF$. Draw DG intersecting AB produced in P. Join PC. Then PCD is the required triangle.



PROOF. Take any other point P' in the line AB. Join $P'D$, $P'C$, and $P'G$. Triangle $P'CA$=triangle $P'GA$ and triangle PCA=triangle PGA. Two sides and included angle being equal in each case. $\therefore P'C=PG$ and $PC=PG$.

Now $P'D+P'G>PD+PG$.

$\therefore P'D+P'C>PD+PC$. Q. E. D.

By passing planes through CD and the points A and B, by a similar construction we may construct a minimum-perimeter triangle upon AB as base with its vertex in CD.

Also solved by *F. R. HONEY*.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.**

Let $x+b(a-x)=cy$ be the equation to EF with AB and AY as axes, and AB, EF the given lines, $AB=a$. Then if $\cot\theta+b\cot\varphi=c$ the vertex C will move on EF. Let $AC=r$, $CB=s$.

$$\text{Then } r+s=\frac{a(\sin\theta+\sin\varphi)}{\sin(\theta+\varphi)}=\text{minimum} \dots\dots\dots\dots (1).$$

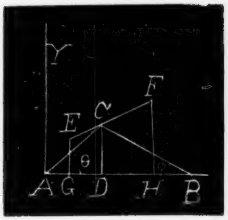
$$\cot\theta+b\cot\varphi=c \dots\dots\dots\dots (2).$$

From (1), $\dfrac{d\theta}{d\varphi}=-\dfrac{\sin\theta}{\sin\varphi}$, from (2), $\dfrac{d\theta}{d\varphi}=-\dfrac{b\sin^2\theta}{\sin^2\varphi}$.

$\therefore \sin\varphi=b\sin\theta$, this in (2) gives

$$\cot\theta=\frac{b^2+c^2-1}{2c},\ \cot\varphi=\frac{c^2-b^2+1}{2bc},\ \cot\theta+\cot\varphi=\frac{AD+DB}{DC}=a/DC,$$

$$\therefore\ DC=\frac{2abc}{b^3-b^2-b+bc^2+c^2+1}.$$



DC must be $> EG$ and $< HF$.

Let $AG=m$, $EG=n$, $AH=h$, $HF=k$.

$$\text{Then } b=\frac{hn-km}{ak+hn-an-km},\quad c=\frac{a(h-m)}{ak+hn-an-km},$$

b must be less than unity or EF may intersect AB.

Let $m=n=1$, $h=a=10$, $k=4$. $\therefore\ b=\frac{1}{6}$, $c=\frac{5}{2}$, $DC=\frac{36}{35}$.

$\cot\varphi=\frac{131}{15}$, $\cot\theta=\frac{19}{18}$. $\therefore\ \theta=43^\circ\ 27'\ 6''$, $\varphi=6^\circ\ 31'\ 56''$.

$AD=DC\cot\theta=\frac{38}{35}$.

89. **Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

Describe a circle tangent to three given circles. [From *Chauvenet's Geometry*, page 318, ex. 213.

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.; and F. R. HONEY, Ph. B., Instructor in Mathematics in Trinity College, New Haven, Conn.

In figure 2 let L, M, N be the circles radii a, b, c.

With M as center and radius $b-a$ describe a circle, also with N as center and radius $c-a$ describe a circle. Draw a circle through L tangent to the circles last described at T, S then the center of this circle is the center of one of the tangent circles. Similarly we can find seven other tangent circles.

Of the eight circles one is tangent to the three circles externally, one is tangent internally, three are tangent to two externally and one internally, and three are tangent to two internally and one externally.

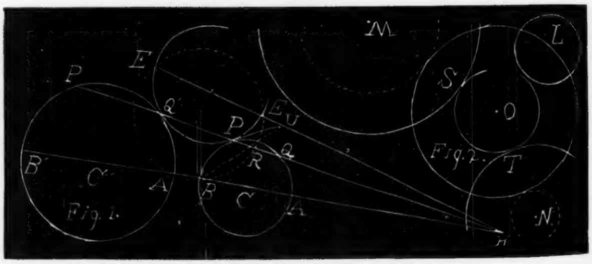


Fig. 1. Fig. 2.

In figure 1, to find a circle passing through a point E and tangent to two circles C, C'. Let H be the point where the external common tangent meets $C'C$ produced. Through $A'BE$ describe a circle cutting EH again in E'. Draw BR meeting HE in U, and draw UP tangent to C, then the circle through PEE' is the circle required.

Two tangents can be drawn from U.

If we had used the point where the internal common tangent cuts CC' we would have determined two other circles, four in all, satisfying the condition.

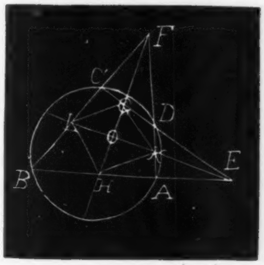
Also solved by *PROF. F. E. MILLER*, and *CHAS. C. CROSS*. Prof. Cooper D. Schmitt did not solve the problem but gave several references where solutions are given. Prof. J. Scheffer gave a short historical note on the problem.

In a future issue of the MONTHLY, we expect to publish a somewhat exhaustive discussion of this very interesting problem.

90. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

The bisectors of the angles of the opposite sides (produced) of an inscribed quadrilateral cut the sides at the angular points of a rhombus.

Solution by G. I. HOPKINS, A. M., Professor of Mathematics in High School, Manchester, N. H.; J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pa.; J. W. SCROGGS, Principal of Rogers Academy, Rogers, Ark.; NELSON L. RORAY, Professor of Mathematics, South Jersey Institute, Bridgeton, N. J.; HENRY N. DAVIS, Providence, R. I.; ALOIS F. KOVARIK, Professor of Mathematics, Decorah Institute, Decorah, Ia.; and the PROPOSER.



In the triangles AEK and LEC, $\angle AEG = \angle LEC$, $\angle EAK = \angle LCE$.

$\therefore$ $\angle EKA = \angle ELC$. $\therefore$ $\angle DKL = \angle CLK$.

$\therefore$ FH is perpendicular to KL at its middle point. Similarly, EL is perpendicular to GH at its middle point.

$\therefore$ In the right triangles KOG, KOH, $KO = KO$, $GO = OH$.

$\therefore$ $KG = KH$. Similarly $KG = GL = LH$.

$\therefore$ $KGLH$ is a rhombus.

This problem was also solved in a similar manner by *E. T. BUSH* and *S. L. ROWAN*, of the Freshman Class of the University of Mississippi; *P. S. BERG*, *W. H. DRANE*, *F. R. HONEY*, *E. R. ROBBINS*, *B. F. SINE*, *J. SCHEFFER*, and *J. F. TRAVIS*.

CALCULUS.

70. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Graduate Fellow in Mathematics in Vanderbilt University, P. O., Lynnville, Tenn.

Prove $$\int_0^\infty \frac{\cos ax}{1+x^{2n}}dx = -i\frac{\pi}{2n}\sum_{r=1}^{n} \omega^{2r-1} e^{ai\omega^{2r-1}}$$

where n is an integer, a is positive, and ω is $e^{i(\pi/2n)}$.

Solution by the PROPOSER.

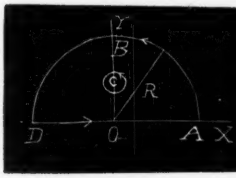
Consider the integral $\int \frac{e^{iay}}{1+y^{2n}}dy$, where a is real and positive. The poles given by $y^{2n} = -1$ or $z = i^{1/n} = \cos(\pi/2n) + i\sin(\pi/2n) = e^{i(\pi/2n)} = \omega$ (say).

It is evident that all the roots of $y^n = i$ are given by ω^{2r-1}, where r may have the values 1, 2, 3, ...n.

Hence $y=\omega^{2r-1}$. About the origin O as center describe a semi-circle ABD with a very large radius, limited by the axis of X (see figure.) About the poles c, which correspond to the points $y^n=i$, describe circles with a very small radius ρ. The proposed function being holomorphic in the portion of the plane lying between the circumferences ρ and the contour $ABDA$, the integrals

$$\int \frac{e^{iay}}{1+y^{2n}}dy$$

relative to the circles and the boundary $ABDA$ are equal. For points on the circles ρ $y=\omega^{2r-1}+\rho e^{i\theta}$, and the integral becomes



$$\int_0^{2\pi} \frac{e^{ia(\omega^{2r-1}+\rho e^{i\theta})}}{1+(\omega^{2r-1}+\rho e^{i\theta})} i\rho e^{i\theta}d\theta, = i\int_0^{2\pi} \frac{e^{ai\omega^{2r-1}}}{2n\omega^{(2r-1)(2n-1)}}d\theta$$

(when ρ becomes infinitesimal),

$$= -i\int_0^{2\pi} \frac{\omega^{2r-1}e^{ai\omega^{2r-1}}}{2n}d\theta, = -i\frac{\pi}{n}\sum_{r=1}^{n}\omega^{2r-1}e^{ai\omega^{2r-1}}.$$

For points on the semi-circle ABD, $y=R\cos\theta+iR\sin\theta$, and the integral becomes,

$$i\int_0^{\pi} \frac{Re^{i\theta}e^{aR(\cos\theta+i\sin\theta)}}{1+R^{2n}e^{i2n\theta}}d\theta,$$

which is evidently equal to zero when $R=\infty$, and we have left the integral along DA, which is

$$\int_{-\infty}^{\infty} \frac{e^{aix}}{1+x^{2n}}dx = -i\frac{\pi}{n}\sum_{r=1}^{n}\omega^{2r-1}e^{ai\omega^{2r-1}} = \int_{-\infty}^{0}\frac{e^{aix}dx}{1+x^{2n}} + \int_0^{\infty}\frac{e^{aix}dx}{1+x^{2n}}$$

$$= \int_0^{\infty}\frac{e^{aix}+e^{-aix}}{1+x^{2n}}dx, = 2\int_0^{\infty}\frac{\cos ax}{1+x^{2n}}dx.$$

Therefore $\displaystyle\int_0^{\infty}\frac{\cos ax\,dx}{1+x^{2n}} = -i\frac{\pi}{2n}\sum_{r=1}^{n}\omega^{2r-1}e^{ai\omega^{2r-1}}$.

Is this result correst? Forsyth gives, on page 41 of his *Theory of Functions*, the integral

$$\int_{-\infty}^{\infty}\frac{\cos ax\,dx}{1+x^{2n}} = -i\frac{\pi}{2n}\sum_{r=1}^{n}\omega^{2r-1}e^{ai\omega^{2r-1}}.$$

[Prof. Zerr remarks that the result is correct, as is easily seen from the following:

Let $\cos ax/(1+x^{2n})=f(x)$.

$$\int_{-\infty}^{\infty} f(x)dx=\int_{0}^{\infty} f(x)dx+\int_{-\infty}^{0} f(x)dx=\int_{0}^{\infty} f(x)dx-\int_{\infty}^{0} f(x)dx=\int_{0}^{\infty} f(x)dx$$

$$+\int_{0}^{\infty} f(x)dx=2\int_{0}^{\infty} f(x)dx.$$

$$\therefore \int_{0}^{\infty} f(x)dx=\tfrac{1}{2}\int_{-\infty}^{\infty} f(x)dx,=-i\frac{\pi}{2n}\sum_{r=1}^{n} \omega^{2r-1}e^{ai\omega^{2r-1}}.]$$

71. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Form the differential equation of the third order, of which $y=c_1e^{2x}+c_2e^{-3x}+c_3e^{x}$ is the complete primitive.

I. Solution by EDGAR ODELL LOVETT, Ph. D., Princeton University, Princeton, N. J.

1°. This problem is a familiar one to students of differential equations. The original primitive together with the results of three successive differentiations, may be written

$$y-e^{2x}c_1-e^{-3x}c_2-e^{x}c_3=0,$$
$$y'-2e^{2x}c_1+3e^{-3x}c_2-e^{x}c_3=0,$$
$$y''-4e^{2x}c_1-9e^{-3x}c_2-e^{x}c_3=0,$$
$$y'''-8e^{2x}c_1+27e^{-3x}c_2-e^{x}c_3=0;$$

where $y'\equiv\dfrac{dy}{dx}$, $y''\equiv\dfrac{d^2y}{dx^2}$, $y'''\equiv\dfrac{d^3y}{dx^3}$.

The above is a system of linear and homogeneous equations in the quantities 1, $e^{2x}c_1$, $e^{-3x}c_2$, and $e^{x}c_3$, hence the determinant of their coefficients vanishes, that is

$$\begin{vmatrix} y & 1 & 1 & 1 \\ y' & 2 & -3 & 1 \\ y'' & 4 & 9 & 1 \\ y''' & 8 & -27 & 1 \end{vmatrix} \equiv \begin{vmatrix} y & 1 & 1 & 1 \\ y'-y & 1 & -4 & 0 \\ y''-y & 3 & 8 & 0 \\ y'''-y & 7 & -28 & 0 \end{vmatrix} \equiv 4 \begin{vmatrix} y'-y & 1 & 1 \\ y''-y & 3 & -2 \\ y'''-y & 7 & 7 \end{vmatrix} \equiv 0;$$

whence

$$\begin{vmatrix} y'-y & 1 & 0 \\ y''-y & 3 & -2 \\ y'''-y & 7 & 0 \end{vmatrix} \equiv 2 \begin{vmatrix} y'-y & 1 \\ y'''-y & 7 \end{vmatrix} = 0;$$

or finally

$$y'''-7y'+6y=0$$

is the differential equation of the third order whose complete primitive is

$$y-ae^{2x}-be^{-3x}-ce^{x}=0.$$

2°. If the problem be generalized and the complete primitive taken in the form

$$y-ae^{mx}-be^{-(m+n)x}-ce^{nx}=0,$$

the corresponding differential equation of the third order is readily found to be

$$y'''-(m^2+mn+n^2)y'+mn(m+n)y=0.$$

The values $m=2$ and $n=1$ give the original problem.

3°. If the problem be completely generalized and the original primitive taken in the form

$$y-ae^{px}-be^{qx}-ce^{rx}=0,$$

the differential equation is

$$y'''-(p+q+r)y''+(pq+qr+rp)y'-pqry=0.$$

Putting $p+q+r=0$ we have the second case above. If in addition to $p+q+r=0$, $p=2$ and $q=1$, the first particular case appears again.

II. Solution by WALTER HUGH DRANE, A. M., Professor of Mathematics, Jefferson Military College, Washington, Miss.

(1) $y=c_1e^{2x}+c_2e^{-3x}+c_3e^{x}$. Differentiate (1).

(2) $\dfrac{dy}{dx}=2c_1e^{2x}-3c_2e^{-3x}+c_3e^{x}$. Subtract (1) from (2).

(3) $\dfrac{dy}{dx}-y=c_1e^{2x}-4c_2e^{-3x}$. Differentiate (3).

(4) $\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}=2c_1e^{2x}+12c_2e^{-3x}$. Subtract twice (3) from (4).

(5) $\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+2y=20c_2e^{-3x}$. Differentiate (5).

(6) $\dfrac{d^3y}{dx^3}-3\dfrac{d^2y}{dx^2}+2\dfrac{dy}{dx}=-60c_2e^{-3x}$. Add 3 times (5) to (6).

(7) $\dfrac{d^3y}{dx^3}-7\dfrac{dy}{dx}+6y=0$. Q. E. D.

See Johnson's Differential Equations, page 104, example 7.

MECHANICS.

61. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

A body is suspended from a fixed point by an elastic string, which is stretched to double its natural length when the body is in equilibrium. Find how much the body must be depressed, so that when let go, it may just reach the point of suspension.

I. Solution by the PROPOSER.

Let a', a be the stretched and unstretched lengths of the string. T=the tension, λ=the modulus of elasticity, W=the weight attached, and x=the distance of the latter from the point of suspension at any time t from the beginning of motion.

By Hooke's law, $a'=a[1+(T/\lambda)]$ ....(1).

By the problem, when $a'=2a$, $T=W$, and (1) gives $\lambda=W$.

The forces acting are W and T acting downward and upward ; then

$$\frac{W}{g}\frac{d^2x}{dt^2}=W-\frac{W(x-a)}{a} \quad \text{..................(2).}$$

Multiplying both sides of (2) by $2(dx/dt)$ and integrating,

$$\frac{dx^2}{dt^2}=\frac{g}{a}(4ax-x^2)+C \quad \text{....................(3).}$$

When $x=x'$, $(dx/dt)=0$, and (3) gives $C=-(g/a)(4ax'-x'^2)$, and (3) is

$$\frac{dx^2}{dt^2}=\frac{g}{a}[(4ax-x^2)-(4ax'-x'^2)] \quad \text{..................(4).}$$

When the body next comes to rest, $(dx/dt)=0$, and $x=0$, giving $4ax'-x'^2=0$, or $x'=4a$, or $x'=0$. $C=0$ in (3) gives

$$\sqrt{\left(\frac{g}{a}\right)}dt=\frac{dx}{\sqrt{4ax-x^2}} \quad \text{.....................(5).}$$

Integrating, $t=\left[\sqrt{\left(\frac{a}{g}\right)}\text{versin}^{-1}\left(\frac{x}{2a}\right)\right]_0^{4a}=\pi\sqrt{\left(\frac{a}{g}\right)}$ ....(6),

the *time* for the motion.

II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

In what follows we neglect the weight of the string, assume Hooke's Law, that the tention of the string is proportional to its extension beyond the natural length, holds throughout the motion.

Let W=weight of body, l=natural length of string, a=extension due to weight W, b=extension at the time body begins to rise after depression. x=extension at any time t, T=corresponding tension of string.

Then by Hooke's Law, $T=Wx/a$ ....(1).

The differential equation of motion is

$$m\frac{d^2x}{dt^2}=W-T. \quad \text{But } W=Mg,\ T=Mgx/a. \quad \therefore \frac{d^2x}{dt^2}=(g/a)(a-x) \quad \text{......(2).}$$

$$\left(\frac{dx}{dt}\right)^2=(g/a)(2ax-x^2)+B. \quad \text{When } t=0,\ x=b,\ \frac{dx}{dt}=0,\ B=(g/a)(b^2-2ab).$$

$$\therefore\left(\frac{dx}{dt}\right)^2=(g/a)(2ax-x^2+b^2-2ab)=v^2.$$

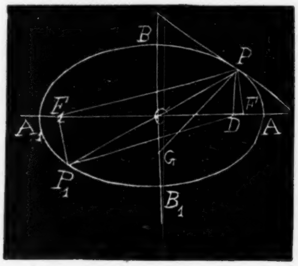
When $x=0$, $v=\sqrt{\frac{gb}{a}}(b-2a)$=velocity of projection, h=height of projection $=\frac{v^2}{2g}=\frac{b}{2a}(b-2a)$, but $h=2l$ and $a=l$.

$\therefore 2l=(b/2l)(b-2l)$ or $b=l(1+\sqrt{5})$.

62. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A particle of mass m moves in the circumference of an ellipse with constant rate v. It is constrained to move in that circumference by attractive forces in the two foci. To determine the magnitude of these forces,

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.



Let P be the particle mass m, $CA=a$, $CB=b$, $PF=r$, $PF_1=r_1$, $CD=x$, $PD=y$, force along $r=f$, force along $r_1=f_1$. R= normal reaction, X=component of f and f_1 parallel to AC, Y=component of f and f_1 parallel to CB, $x^2/a^2+y^2/b^2=1$, the equation to the ellipse. The equations of motion are

$$m(d^2x/dt^2)=X-R(dy/ds)\ ;\ m(d^2y/dt^2)=Y+R(dx/ds)\ldots\ldots(1,2).$$

Now $ds/dt=v$, $\frac{dx}{ds}=\frac{a^2y}{\sqrt{a^4y^2+b^4x^2}}=\frac{ay}{b\sqrt{rr_1}}$, $\frac{dy}{ds}=\frac{b^2x}{\sqrt{a^4y^2+b^4x^2}}=\frac{bx}{a\sqrt{rr_1}}$

$$\left(\frac{dx}{dt}\right)^2=\left(\frac{ds}{dt}\right)^2\frac{a^4y^2}{a^4y^2+b^4x^2}=\frac{a^2v^2y^2}{b^2(a^2-e^2x^2)}=\frac{v^2(a^2-x^2)}{a^2-e^2x^2}.$$

$$\left(\frac{dy}{dt}\right)^2=\left(\frac{ds}{dt}\right)^2\cdot\frac{b^4x^2}{a^4y^2+b^4x^2}=\frac{b^2v^2x^2}{a^2(a^2-e^2x^2)}.$$

$$\therefore\frac{d^2x}{dt^2}=-\frac{a^2v^2x(1-e^2)}{(a^2-e^2x^2)^2}=-\frac{b^2v^2x}{r^2r_1^2},\ \frac{d^2y}{dt^2}=\frac{a^2v^2y}{(a^2-e^2x^2)^2}=\frac{a^2v^2y}{r^2r_1^2}.$$

$$R=(f+f_1)\cos FPG=b^2(f+f_1)/\sqrt{(b^4+a^2e^2y^2)}=(f+f_1)/\sqrt{(rr_1)}.$$

$$X=f_1\cos EPF_1+f\cos EPF=f_1(ae+x)/r_1-f(ae-x)/r.$$

$$Y=f_1\cos DPF_1+f\cos DPF=y(f_1r+fr_1)/rr_1.$$

Substituting in (1) and (2),

$$-ab^2mv^2x=rr_1(arx+a^2er-bx)f_1+rr_1(ar_1x-a^2er_1-bx)f\ \ldots\ldots\ldots\ldots\ldots\ (3).$$

$a^2bmv^2 = rr_1(br+a)f_1 + rr_1(br_1+a)f$ .... (4).

From (3) and (4) since $x=(r_1-r)/2e$,

$$\frac{f_1 r}{a^2r-b^2r_1} = \frac{fr_1}{a^2r_1-b^2r} = \frac{abmv^2}{2[(a^2+b^2+a^2be^2)rr_1-2a^2b^2]}.$$

II. **Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.**

Let ρ=the radius of curvature of the curve at any position, P, of the particle, ϕ=the angle included by either SP or HP, S and H being the foci, and $2a$ =the major axis. Let CD be the semi-diameter conjugate to CP, C being the center of ellipse.

The normal components of the central attractions must together equal the centrifugal force. We may assume the forces in S and H as proportional to some power of CD ; and if the absolute intensities of the two forces are equal, say μ,

$$[\mu(CD)^n + \mu(CD)^n]\cos\phi = v^2/\rho \quad \text{.......... (1).}$$

But $$\rho\cos\phi = \frac{SP.HP}{a} = \frac{CD^2}{a} \quad \text{.......... (2),}$$

and (1) is $$2\mu(CD)^n.\frac{CD^2}{a} = v^2 \quad \text{.......... (3).}$$

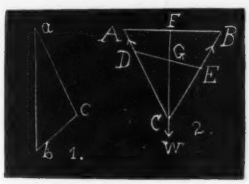
For v=a constant, requires that $n=-2$, showing plainly that the forces vary inversely as the product of the focal distances of the particle.

63. **Proposed by A. H. BELL, HILLSBORO, Ill.**

From a horizontal support at a distance of ten feet apart, a beam 5 feet long and 10 pounds weight is suspended by ropes attached to each end. The ropes are 3 and 5 feet respectively, in length. Required the angles made by the ropes and horizontal support. Also the stress upon each rope.

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Regarding the bar DE as uniform so that the middle point is the center of gravity, we must have, for equilibrium, the three forces AD, BE, GC passing through the same point C, with GC perpendicular to AB.



The $\angle ABC = \angle CDE$, and the $\angle BAC = \angle CED$.

Let $\angle ABC = \theta$, $\angle BAC = \varphi$, stress on $BE=R$, on $AD=R_1$, weight of $DE=W=10$ pounds. Also $AB=10$, $BE=ED=5$, $AD=3$, $AE^2 = 100+25-100\cos\theta = 25+9+30\cos CDE$.

$\therefore$ $130\cos\theta=91$. $\therefore$ $\cos\theta=\frac{91}{130}=.70000$. $\therefore$ $\theta=45°\ 34'\ 23''$.

$DB^2=100+9-60\cos\varphi=25+25+50\cos CED$.

$\therefore$ $110\cos\varphi=59$. $\therefore$ $\cos\varphi=\frac{59}{110}=.53636$. $\therefore$ $\varphi=57°\ 33'\ 50''$.

Let figure 2 represent the force diagram, $ab=W$, $bc=R$, $ac=R_1$, $\angle abc=90^\circ-\theta$, $\angle bac=90^\circ-\varphi$, $\angle acb=\theta+\varphi$.

$\therefore R=W\cos\varphi/\sin(\theta+\varphi)$, $R_1=W\cos\theta/\sin(\theta+\varphi)$.

$\therefore R=5.5078$ pounds, $R_1=7.1881$ pounds.

DIOPHANTINE ANALYSIS.

61. Proposed by SYLVESTER ROBBINS, North Branch Depot, New Jersey.

Investigate that infinite series of prime, integral, rational scalene triangles where the sides of every term are consecutive numbers; then take the necessary factors from the proper KEY, and by an expeditious method, find in their order the areas of ten initial terms.

Solution by the PROPOSER.

I. The KEY to this series of rational triangles is $\sqrt{3}=\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \frac{265}{153}, \frac{362}{209}, \frac{989}{571}, \frac{1351}{780}, \frac{3691}{2131}, \frac{5042}{2911}, \frac{13775}{7953}, \frac{18817}{10864}, \frac{51409}{29681}, \frac{70226}{40545}, \frac{191861}{110771}, \frac{262087}{151316}$, etc. Regard the mean side as the base, and drop perpendicular from the opposite angle. Let $x=\frac{1}{2}$base. Notice that $x-2$ and $x+2$ are the segments of the base, and $\sqrt{[3(x^2-1^2)]}$ is the altitude of the triangle. Find such values for x as will render $\sqrt{[3(x^2-1^2)]}$ rational.

When $x=1$, 2, 7, 26, 97, 362, 1351, 5042, 18817, 70226, 262087, $\sqrt{[3(x^2-1^2)]}=0$, 3, 12, 45, 168, 627, 2320, 8733, 32592, 121635, 453948, etc.

These values of x are the half-bases of the several triangles. They are also the numerators of the even convergents in the expansion of $\sqrt{3}$. The values of $\sqrt{[3(x^2-1^2)]}$ are the altitudes of the same triangles, respectively, and they are also three times the denominators of the even convergents in the expansion of $\sqrt{3}$. Multiply one-half the base of a triangle by its perpendicular height, or, three times the product of the terms of the nth even convergent, must give the area of the nth triangle in the series.

Thus, $3\times2\times1=6$; $3\times7\times4=84$; $3\times26\times15=1170$; $3\times97\times56=16296$; $3\times362\times209=226974$; $3\times1351\times780=3161340$; $3\times5042\times2911=44031786$; $3\times18817\times10864=613283664$; $3\times70226\times40545=8541939510$; $3\times262087\times151316=118973869476$, etc.

II. Numerators of even convergents in expansion of $\sqrt{3}$: 1, 2, 7, 26, 97, 362, 1351, 5042, 18817, 70226, etc. Then $\frac{1}{8}(7^2-1^2)=6$; $\frac{1}{8}(26^2-2^2)=84$; $\frac{1}{8}(97^2-7^2)=1170$; $\frac{1}{8}(362^2-26^2)=16296$; $\frac{1}{8}(1351^2-97^2)=226974$; etc.

III. Denominators of even convergents : 1, 4, 15, 56, 209, 780, 2911, etc. $\frac{3}{8}(4^2-0)=6$; $\frac{3}{8}(15^2-1^2)=84$; $\frac{3}{8}(56^2-4^2)=1170$; $\frac{3}{8}(209^2-15^2)=16296$; $\frac{3}{8}(780^2-56^2)=226974$; etc.

IV. Let x=the half-sum of the three sides of the triangle. Then $\frac{1}{3}x-1$, $\frac{1}{3}x$ and $\frac{1}{3}x+1$ are the remainders.

$(x)[(\frac{1}{3}x)-1](\frac{1}{3}x)[(\frac{1}{3}x)+1]$=square of triangle. $3(x^2-3^2)$=square.

$\sqrt{\{(\frac{1}{3}x^2)[(x^2-3^2)/9]\}}/x=\frac{1}{3}\sqrt{[(x^2-3^2)/3]}$, the radius of inscribed circle.

Put $x=y+6$; then $3[(y+6)^2-3^2]$=square=$(my+9)^2$.

$3y^2+36y+81=m^2y^2+18my+81$; $3y+36=m^2y+18m$. $y=(18m-36)/(3-m^2)$; and $x=y+6=(18m-18-6m^2)/(3-m^2)$.

Say $m=\sqrt{3}=\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}; \frac{26}{15}$, etc. Substitute these values of m in the formula, and we obtain, when

$m=1, x=3$, and $\frac{1}{3}\sqrt{[x^2-3^2)/3]}=0$;
$m=2, x=6$, " " $=1$;
$m=\frac{5}{3}, x=21$, " " $=4$;
$m=\frac{7}{4}, x=78$, " " $=15$;
$m=\frac{19}{11}, x=291$, " " $=56$;
$m=\frac{26}{15}, x=1086$. " " $=209$; etc.

Here the several values of x, the half-sums of three sides, are also three times the numerators of the even convergents in the expansion of $\sqrt{3}$, and the several values of $\frac{1}{3}\sqrt{[(x^2-3^2)/3]}$ are the radii of the inscribed circles, respectively, and also the denominators of the even convergents. Multiplying the half-sum by the radius of the inscribed circle, or taking three times the product of the terms of the nth even convergent in the expansion of $\sqrt{3}$ will give the area of the nth triangle in the series where the common difference in sides is unity.

V. Denominators of odd convergents : 1, 3, 11, 41, 153, 571, 2131, etc. $\frac{3}{4}(3^2-1^2)=6$; $\frac{3}{4}(11^2-3^2)=84$; $\frac{3}{4}(41^2-11^2)=1170$; $\frac{3}{4}(153^2-41^2)=16296$; $\frac{3}{4}(571^2-153^2)=226974$; etc.

VI. Numerators of odd convergents : 1, 5, 19, 71, 265, 989, 3691, etc. $(2\frac{1}{2})^2-(\frac{1}{2})^2=6$; $(9\frac{1}{2})^2-(2\frac{1}{2})^2=84$; $(35\frac{1}{2})^2-(9\frac{1}{2})^2=1170$; $(132\frac{1}{2})^2-(35\frac{1}{2})^2=16296$; $(494\frac{1}{2})^2-(132\frac{1}{2})^2=226974$; etc.

VII. Let $x+1$, x and $x-1$ be the remainders. Then $3x$=the half-sum. $(3x)(x+1)(x)(x-1)$=square of area. $3(x^2-1^2)$=square. Put $x=y+2$. $3[(y+2)^2-1^2]$=square$=(my+3)^2$. $3y^2+12y+9=m^2y^2+6my+9$; $y=(6m-12)/(3-m^2)$; $x=(6m-6-2m^2)/(3-m^2)$. Consent that $m=\sqrt{3}=\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}$, etc. Substitute in formula, and $x=1, 2, 7, 26, 97, 362, 1351$, etc., and the areas of the triangles are as follows : $\sqrt{(3\times2\times1\times0)}=3\times1\times0=0$; $\sqrt{(6\times3\times2\times1)}=3\times2\times1=6$; $\sqrt{(21\times8\times7\times6)}=3\times7\times4=84$; $\sqrt{(18\times27\times26\times25)}=3\times26\times15=1170$; $\sqrt{(291\times98\times97\times96)}=3\times97\times56=16296$; $\sqrt{(1086\times363\times362\times361}=3\times362\times209=226974$, etc.

VIII. Odd convergents : $\sqrt{3}=\frac{1}{1}, \frac{5}{3}, \frac{19}{11}, \frac{71}{41}. \frac{265}{153}$, etc. $6\times1\times1=6$; $(6\times3\times5)-6=84$; $(6\times11\times19)-84=1170$; $(6\times41\times71)-1170=16296$; $(6\times153\times265)-16296=226974$; etc.

IX. Relation of areas : $14\triangle_n-\triangle_{n-1}=\triangle_{n+1}$; $M=14$. 6 ; $14\times6=84$; $(14\times84)-6=1170$; $(14\times1170)-84=16296$; $(14\times16296)-1170=226974$, etc.

X. When x represents the half-base, $2x-1$, $2x$ and $2r+1$ are the sides, $x-2$ and $x+2$ are the segments of the base , $\sqrt{[3(x^2-1^2)}$ is the altitude of the triangle ; and $x\sqrt{[3(x^2-1^2)]}/(4x^2-1^2)$ is the half-sine of angle opposite base. Giving x same values as in I, the half-sines are found to be $\frac{2}{5}=(2\times1)/(2^2+1^2)$; $\frac{28}{65}=(7\times4)/(7^2+4^2)$; $\frac{390}{901}=(26\times15)/(26^2+15^2)$; $\frac{5432}{12545}=(97\times56)/(97^2+56^2)$; $75658/(362^2+209^2)=(362\times209)/(362^2+209^2)$; etc.

Multiply the product of two sides by one-half sine of included angle for area of triangle.

$[(2\times2)-1][(2\times2)+1][(2\times1)/(2^2+1^2)]=3\times5\times\frac{2}{5}=3\times1\times2=6$; $[(2\times7)-1]$

$[(2\times7)+1][(4\times7)/(4^2+7^2)]=13\times15\times\frac{28}{65}=3\times4\times7=84$; $[(2\times26)-1][(2\times26+1]$ $[(15\times26)/(15^2+26^2)]=51\times53\times\frac{390}{901}=3\times15\times26=1170$; $[(2\times97)-1][(2\times97+1]$ $[(56\times97)/(56^2+97^2)]=193\times195\times\frac{5432}{12545}=3\times56\times97=16296$; $[(2\times362)-1][(2\times 362)+1][(209\times362)/(209^2+362^2)]=723\times725\times\frac{75658}{174725}=3\times209\times362=226974$; etc.

Here it should be noticed that in canceling both sides the denominator of the half-sine disappears, and three times the product of the terms of the nth even convergent in the expansion of $\sqrt{3}$ brings the area to light; also observe, since sides and denominator fall out of view, and factor 3 stands constant, the area must be determined by the numerator of these half-sines, and *this* series may be continued by use of Magic $M=14$; $14\times2=28$; $(14\times28)-2=390$; $(14\times390)-28=5432$; $(14\times5432)-390=75658$; etc.

Also solved by *JOSIAH H. DRUMMOND*, *M. A. GRUBER*, and *G. B. M. ZERR*.

AVERAGE AND PROBABILITY.

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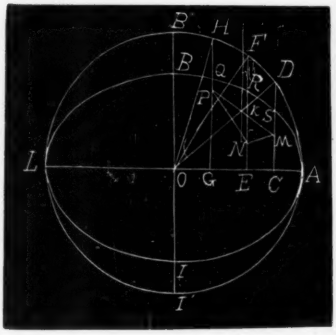
60. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Four points are taken at random within an ellipse. What is the chance that they form a reentrant quadrilateral?

Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

We will solve this problem for the quadrant, the semi-ellipse, and the whole ellipse.

Let $ABLI$ be the ellipse, and $AB'LI'$ the circumscribing circle; M, N, P the three random points; through M, N, P draw CD, EF, GH perpendicular to AO, EF intersecting MP at K. The triangle will pass through all the possible variations by considering only those relative positions of the points in which CD lies to the right of GH, and EF between CD and GH.



If the fourth point falls anywhere on the triangle formed by joining the points M, N, P, the quadrilateral thus formed will be reentrant.

Let $OA=a$, $OB=b$, $GP=x$, $CM=y$, $EN=z$, $GQ=x'$, $CS=y'$, $ER=z'$, $EK=z''$, $\angle GOH=\theta$, $\angle COD=\phi$, $\angle EOF=\psi$.

Then we have $x'=b\sin\theta$, $y'=b\sin\phi$, $z'=b\sin\psi$, $v=1/(\cos\phi-\cos\theta)$, $z''=v[x(\cos\phi-\cos\psi)+y(\cos\psi-\cos\theta)]$.

Area $MNP=\frac{1}{2}a[x(\cos\phi-\cos\psi)+y(\cos\psi-\cos\theta)+z(\cos\theta-\cos\phi)]=u$, when $z<z''$. Area $MNP=\frac{1}{2}a[x(\cos\psi-\cos\phi)+y(\cos\theta-\cos\psi)+z(\cos\phi-\cos\theta)]=u_1$, when

$z>z''$. An element of surface at M is $a\sin\varphi d\varphi dy$, at N it is $a\sin\psi d\psi dz$, at P it is $a\sin\theta d\theta dx$.

The limits of θ are (for quadrant) 0 and $\frac{1}{2}\pi$; of φ, 0 and θ; of ψ, φ and θ; of x, 0 and x'; of y, 0 and y'; of z, 0 and z'', and z'' and z'.

Hence the required average area is,

$$\Delta=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}\int_0^{y'}\left(\int_0^{z''}udz+\int_{z''}^{z'}u_1dz\right)a\sin\theta d\theta a\sin\varphi d\varphi a\sin\psi d\psi dxdy}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}\int_0^{y'}\int_0^{z'}a\sin\theta d\theta a\sin\varphi d\varphi a\sin\psi d\psi dxdydz}$$

$$=\frac{384}{\pi^3b^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}\int_0^{y'}\left(\int_0^{z''}udz+\int_{z''}^{z'}u_1dz\right)\sin\theta\sin\varphi\sin\psi d\theta d\varphi d\psi dxdy$$

$$=\frac{96a}{\pi^3b^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}\int_0^{y}\{[x(\cos\varphi-\cos\psi)+y(\cos\psi-\cos\theta)]^2+[x(\cos\varphi-\cos\psi)$$

$$+y(\cos\psi-\cos\theta)+b\sin\psi(\cos\theta-\cos\psi)]^2\}\sin\theta\sin\varphi\sin\psi v d\theta d\varphi d\psi dxdy$$

$$=\frac{32a}{\pi^3b^2}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}\int_0^{x'}[6x^2\sin\varphi(\cos\varphi-\cos\psi)^2+6bx\sin^2\varphi(\cos\varphi-\cos\psi)(\cos\psi$$

$$-\cos\theta)+6bx\sin\varphi\sin\psi(\cos\varphi-\cos\psi)(\cos\theta-\cos\varphi)+2b^2\sin^3\varphi(\cos\psi-\cos\theta)^2$$

$$+3b^2\sin\varphi\sin^2\psi(\cos\theta-\cos\varphi)^2+3b^2\sin^2\varphi\sin\psi(\cos\theta-\cos\varphi)(\cos\psi-\cos\theta)]$$

$$\times\sin\theta\sin\varphi\sin\psi v d\theta d\varphi d\psi dx.$$

$$\Delta=\frac{32ab}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_{\Phi}^{\theta}[2\sin^3\theta\sin\varphi(\cos\varphi-\cos\psi)^2+2\sin\theta\sin^3\varphi(\cos\psi-\cos\theta)^2$$

$$+3\sin^2\theta\sin^2\varphi(\cos\varphi-\cos\psi)(\cos\psi-\cos\theta)+3\sin\theta\sin\varphi\sin^2\psi(\cos\theta-\cos\varphi)^2$$

$$+3\sin^2\theta\sin\varphi\sin\psi(\cos\varphi-\cos\psi)(\cos\theta-\cos\varphi)$$

$$+3\sin\theta\sin^2\varphi\sin\psi(\cos\psi-\cos\theta)(\cos\theta-\cos\varphi)]\sin\theta\sin\varphi\sin\psi v d\theta d\varphi d\psi$$

$$=\frac{16ab}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}[4\sin^2\theta\cos^2\varphi+4\sin^2\varphi\cos^2\varphi+4\sin^2\theta\cos^2\theta+4\sin^2\varphi\cos^2\theta$$

$$+\sin^2\theta\cos\theta\cos\varphi+\sin^2\varphi\cos\varphi\cos\theta-6\sin\theta\cos\theta\sin\varphi\cos\varphi$$

$$+6\cos^3\theta\cos\varphi+6\cos\theta\cos^3\varphi+12+6\cos^2\theta+6\cos^2\varphi-36\cos\theta\cos\varphi$$

$$-12\sin\theta\sin\varphi-9(\theta-\phi)\sin\theta\cos\varphi+9(\theta-\phi)\sin\varphi\cos\theta]\sin^2\theta\sin^2\varphi d\theta d\varphi$$

$$=\frac{8ab}{9\pi^3}\int_0^{\frac{1}{2}\pi}(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta-45\sin\theta\cos\theta$$

$$-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta)\sin^2\theta d\theta$$

$$=\frac{ab}{\pi}\left(\frac{35}{12}+\frac{16}{3\pi}-\frac{131}{3\pi^2}\right).$$

For the semi-ellipse above the major axis, the limits of θ are 0 and π, and those of the other variables the same as above. The number of ways the three points can be taken in the semi-ellipse is eight times the number of ways in a quadrant, and hence we get

$$\triangle_1=\frac{ab}{9\pi^3}\int_0^{\pi}(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta-45\sin\theta\cos\theta$$

$$-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta)\sin^2\theta d\theta=\frac{ab}{\pi}\left(\frac{35}{24}-\frac{32}{3\pi^2}\right).$$

For the limits of θ are 0 and 2π, and the points can be taken eight times the number of ways in semi-ellipse. Hence

$$\triangle_2=\frac{ab}{72\pi^3}\int_0^{2\pi}(69\theta+36\theta\cos\theta-12\theta\sin^2\theta-12\theta\sin^4\theta-60\sin\theta-45\sin\theta\cos\theta$$

$$-10\sin^3\theta\cos\theta+3\sin^5\theta\cos\theta)\sin^2\theta d\theta=35ab/48\pi.$$

Let C, C_1, C_2 be the respective chances required.

$$C=\frac{4\triangle}{\pi ab}=\frac{4}{\pi^2}\left(\frac{35}{12}+\frac{16}{3\pi}-\frac{131}{3\pi^2}\right);\quad C_1=\frac{4\triangle_1}{\pi ab}=\left(\frac{35}{42}-\frac{32}{3\pi^2}\right);$$

$$C_2=\frac{4\triangle_2}{\pi ab}=\frac{35}{12\pi^2}.$$

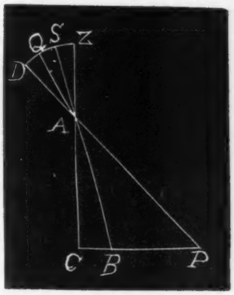
MISCELLANEOUS.

58. Proposed by EDMUND FISH, Hillsboro, Ill.

The longest noonday winter shadow of an upright object is found to be seven times as long as the shortest summer shadow of the same object. Required the latitude of the place.

I. Solution by S. HART WRIGHT, A. M., M. D., Ph. D., Penn Yan, N. Y.

In the right plane triangles ABC and APC, let the vertical AC(=unity) be the rod that casts a shadow from C to B, and from C to P, when the sun is at S and D. Extend CA to the zenith Z, BA to S and PA to D. Bisect DAS with QA. Let $\angle BAC=\chi=ZAS$. DAS is double the obliquity of the ecliptic$=2\delta=\angle PAB=46^\circ\ 54'\ 30''=v$. $DAQ=SAQ=\delta$, and Q must be on the equator, and QAZ=the required latitude$=\lambda$. $CP=7CB$. Put $7=m$, and $\tan^{-1}m=\beta=81^\circ\ 52'\ 12''$. We have $CB=\tan\chi$, and $CP=\tan(\chi+v)$, and $m\tan\chi=\tan(\chi+v)$, a trigonometric equation, from which we derive $\sin(2\chi\pm v)=\cot(\beta\mp45^\circ)\sin v$. Four values of χ result, the upper signs giving the only acceptable value of $\chi=14^\circ\ 57'\ 30''$. The other signs make the $\angle PAC>90^\circ$. Now $\lambda=\chi+\delta=38^\circ\ 24'\ 45''$ north or south, as the seasons are interchangeable on each side of the equator.



II. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

Let AC be the upright object length$=l$, CP the earth, SB the summer sun, DP the winter sun. Let ϕ=latitude, δ=north, $-\delta$=south declination of the sun, and let the winter shadow be n times as long as the summer shadow.

Then $\angle CAB=(\phi-\delta)$, $\angle CAP=(\phi+\delta)$, $CP=n.CB$, $CP=l\tan(\phi+\delta)$, $CB=l\tan(\phi-\delta)$.

$\therefore\ l\tan(\phi+\delta)=nl\tan(\phi-\delta)$.

$\therefore\ (n+1)\tan^2\phi\tan\delta-(n-1)\tan\phi\sec^2\delta+(n+1)\tan\delta=0$.

$$\tan^2\phi-\frac{2(n-1)\tan\phi}{(n+1)\sin2\delta}+1=0.$$

$$\therefore\ \tan\phi=\frac{(n-2)\pm\sqrt{[(n-1)^2-(n+1)^2\sin^2 2\delta]}}{(n+1)\sin2\delta}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(A).$$

Now $\delta=23^\circ\ 27'\ 30''$, $n=7$. $\therefore\ \tan\phi=.793428$ or 1.26035.

$\therefore\ \phi=38^\circ\ 25'\ 46''$ or $51^\circ\ 34'\ 14''$.

The two values of $\tan\phi$ are equal when $n=(1+\sin2\delta)/(1-\sin2\delta)$.

$\therefore\ n=6.4118$, $\phi=45^\circ$. When $\phi=\delta$, and $\phi+\delta=90^\circ$, n is infinite.

In the first case the summer shadow is zero and the winter shadow is finite; in the second case, the winter shadow is infinite and the summer shadow is finite.

In formula (A), δ and n can have any values within proper limits.

Also solved by *W. W. LANDIS*, and *J. SCHEFFER*.

59. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

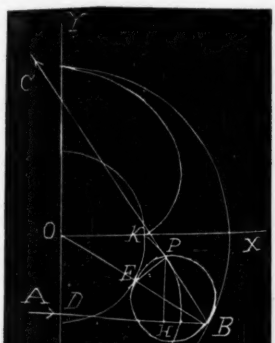
When a cylindrical china jar, standing upon the ground, receives the sun's rays obliquely, a bright curve is observed to form itself at the bottom of the jar, and it is found that the shape and dimensions of this curve are not affected by the varying elevations of the sun: account for this latter circumstance, and determine the nature of the bright curve. [From *Parkinson's Optics*.]

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Mississsppi.

All rays striking any element of the cylindrical surface lie in a vertical plane. Their reflections form the other face of the dihedral angle whose bisector passes through the axis of the cylinder. These reflected rays intersect the base of the cylinder in a straight line. There is thus formed a system of lines, and the bright curve observed is their envelope. The altitude of the sun does not affect the position of the vertical planes; and, therefore, the intersections with the bottom of the jar are unchanged, and the continual intersection of the consecutive lines so formed produces a curve invariable as to its shape and size.

The bright curve is the caustic by reflection for the circle, the incident rays being parallel. The following general property of caustics by reflection for parallel rays is established in *Price's Infinitesimal Calculus*: "The distance from the incident point in the reflecting curve to the point of intersection of two consecutive reflected rays, is equal to one-fourth of the chord of the circle of curvature at the point of incidence which is parallel to the incident ray."

A. Take the center of the circle as the origin, the X-axis parallel to the incident rays, the Y-axis perpendicular to them. Let AB be an incident ray, BC its reflection, the angle between them being 2θ. Take BP along BC equal to one-half of DB, D being the intersection of AB with the Y-axis. Then, according to the principle quoted above, P is a point of the caustic. To find the locus of P: Draw OB, denoting it by a. From P drop a perpendicular to AB meeting it at H. Denoting the coördinates of P by x and y,



$$x=DB-HB=a\cos\theta-\tfrac{1}{2}a\cos\theta\cos2\theta,$$

$$y=OD-PH=a\sin\theta-\tfrac{1}{2}a\cos\theta\sin2\theta.$$

From these $x=\tfrac{3}{4}a\cos\theta-\tfrac{1}{4}a\cos3\theta$,

and $y=\tfrac{3}{4}a\sin\theta-\tfrac{1}{4}a\sin3\theta$.

These may be written

$x=(\frac{1}{2}a+\frac{1}{4}a)\cos\theta-(\frac{1}{4}a)\cos[(\frac{1}{2}a+\frac{1}{4}a)/(\frac{1}{4}a)]\theta$,

and $y=(\frac{1}{2}a+\frac{1}{4}a)\sin\theta-(\frac{1}{4}a)\sin[(\frac{1}{2}a+\frac{1}{4}a)/(\frac{1}{4}a)]\theta$.

These are the well-known equations of an epicycloid, the radii of the fixed and rolling circles being $\frac{1}{2}a$ and $\frac{1}{4}a$ respectively.

B. The following geometrical solution is very much like one given in *Wood's Optics*, and was suggested by it.

Referring to the same figure, erect at P a perpendicular to CB, meeting OB at E. Comparing the similar triangles EPB and ODB, $ED : DB=BP : BD=1 : 2$.

If, then, upon EB, the half of OB, as diameter, a circumference be drawn its intersection with CB will be a point of the caustic. With O as center and EO as radius describe a semi-circle, intersecting the X-axis at K.

The $\angle EOK=\theta$, and arc $EK=\frac{1}{2}a\theta$.

Also, since $\angle EBP=\theta$, the angle at the center measured by arc $EP=2\theta$; and arc $EP=\frac{1}{4}a.2\theta=\frac{1}{2}a.\theta$.

Hence arc EP=arc EK.

The locus of P is, therefore, generated by the circle EPB rolling on the circle EK, the points P and K being originally in contact.

Of course the problem may be solved without assuming the property quoted from *Price*. In *Rice and Johnson's Differential Calculus* an excellent solution is outlined.

Also solved by *C. W. M. BLACK*, *S. H. WRIGHT*, and *B. F. FINKEL*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

97. Proposed by J. A. CALDERHEAD, M. Sc., Professor of Mathematics, Curry University, Pittsburg, Pa.

In what time will \$4000 amount to \$5134.96, interest at 6% payable annually ?

*** Solutions of these problems should be sent to B. F. Finkel, not later than July 10.

GEOMETRY.

97. Proposed by CHAS. C. CROSS, Libertytown, Md.

Prove by pure geometry: The radius of a circle drawn through the centers of the inscribed and any two escribed circles of a triangle is double the radius of the circumscribed circle of the triangle.

98. Proposed by EDW. R. ROBBINS, Master in Mathematics and Physics, Lawrenceville School, Lawrenceville, N. J.

Construct a circle which shall pass through two given points and touch a given circle, (1) when the distance between the points is less than the diameter of the circle, and (2) when it is greater.

*** Solutions of these problems should be sent to B. F. Finkel, not later than July 10.

MECHANICS.

69. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A rough sphere of radius a and radius of gyration K, capable of rotating about its center, is initially at rest; another sphere of $1/n$ the mass and of radius b, and radius of gyration k, is placed gently on it, having initially an angular velocity ω about the common normal which makes an acute angle α with the vertical drawn upwards. Prove that the second sphere will not roll off provided

$$\omega^2 > \frac{2\mu(a+b)g}{(3\mu+1)b^2}[(3\mu+1)^2-4\mu^2\cos^2\alpha]\sec\alpha, \text{ where } \mu=a^2/nK^2+b^2/k^2.$$

[From *Routh's Rigid Dynamics.*]

70. Proposed by CHAS. E. MEYERS, Canton, Ohio.

A homogeneous sphere, radius r, having an angular velocity ω, gradually contracts by cooling. What will be the angular velocity at the instant the radius becomes $\frac{1}{2}r$?

*** Solutions of these problems should be sent to B. F. Finkel, not later than August 10.

DIOPHANTINE ANALYSIS.

68. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find a *general* value for p in the expression $4p+1$=the sum of two squares.

69. Proposed by JOSIAH H. DRUMMOND, LL. D., Counselor at Law, Portland, Me.

Two right angled triangles have the same base which is a mean proportional between the two perpendiculars: find a general solution, that will give integral values for all the sides of both triangles.

70. Proposed by PROF. CHARLES CARROLL CROSS, Libertytown, Md.

Give methods for decomposing numbers into squares, cubes, or biquadrates and show that 61×200^3 is the sum of ten cube numbers and that 844933 is the sum of eleven biquadrates in thirteen different ways. [From *The Mathematical Magazine*, Vol. II, No. 10.]

*** Solutions of these problems should be sent to J. M. Colaw, not later than August 10.

AVERAGE AND PROBABILITY.

65. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

What is the average rate of the sun's motion in declination from the equator to the solstices?

66. Proposed by REV. W. ALLEN WHITWORTH, A. M.

A rod 9 feet long is to be divided into three parts, of which A is to have the largest, B the next, and C the smallest. If the two fractures are made at random, A's, B's, and C's expectations will be respectively 66, 30, and 12 inches. But, if one fracture be made at random and the larger portion of the rod be then divided at random, their expectations will be 64, 31, and 13 inches.

67. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A person writes n letters and addresses n envelopes; if the letters are placed in the envelopes at random, what is the probability that every letter goes wrong? [From *Hall and Knight's Higher Algebra.*]

*** Solutions of these problems should be sent to B. F. Finkel, not later than August 10.

MISCELLANEOUS.

62. Proposed by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics, The Russell College, Lebanon, Va.

How many bushels of wheat will a conical bin 8 feet in diameter at base and 12 feet high, hold, if part of the bin is cut off by a plane parallel to the side and passing through the center of the base?

63. Proposed by F. P. MATZ, D. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Show that the path of a projectile moving with a constant velocity is an inverted catenary of equal strength.

*** Solutions of these problems should be sent to J. M. Colaw, not later than August 10.

EDITORIALS.

The degree of Civil Engineer has been conferred on our valued contributor, Fremont Crane, by the University of Minnesota.

In our next issue will appear an article on Symmetric Functions, by Professor E. D. Roe, Associate Professor of Mathematics in Oberlin College, now at Erlangen, Germany.

Dr. David Eugene Smith, Professor of Mathematics in the Michigan State Normal School, has accepted the Presidency of the New York State Normal School, at Rockport.

In our last issue, the last two pages of the MONTHLY were hurried into print without our having read the proof. That we might have the opportunity of eliminating some of the errors which thus appeared in them, we have had those two pages reprinted and bound in the present issue. This, we are sure, will be appreciated by those who wish to have the volumes of the MONTHLY bound.

THE UNIVERSITY OF CHICAGO. During the summer quarter (July 1 to September 23, 1898) the following mathematical courses (four or five hours weekly) will be offered:—By Associate Professor Maschke: Theory of Invariants; Functions of a Complex Variable;—By Assistant Professor Young: Mathematical Pedagogy (to August 15); Culture Calculus; Plane Trigonometry (to August